Green's Function Method for Creating Accurate Stereo Sound Images: Cylindrical Volumes

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Herein is developed a solution for cylindrical volumes for use in the Green's function method described in [1].

Consider the formal solution derived in [1] for the inhomogeneous wave equation, with homogeneous boundary and initial conditions:

$$\psi(\mathbf{r},t) = c^2 \sum_n {}' \left[\Phi_n(\mathbf{r}) \int_V u(\mathbf{r'})\ \Phi_n^*(\mathbf{r'}) \int_{t'=t_0}^{t+} s(t') \frac{\sin(\omega_n \tau)}{\omega_n}\ \theta(\tau)\ dt'\, dV' \right]$$

The product $u(\mathbf{r'})\ s(t')$ comprises the source term $\rho(\mathbf{r'}, t')$ for an audio signal; t_0 and $t+$ are the start and current times for the signal source, respectively, where we consider quiescent initial conditions; $t+ = t + \varepsilon$ and we will take the limit of $t+$ as $\varepsilon \to 0$ at the end of the calculation, once again following [2]. $\theta(t)$ is the Heaviside step function.

Once again the key to efficiently solving a number of important technological problems is that for certain cases it is possible to integrate, relatively easily and prior to computation, the integral:

$$\int_V u(\mathbf{r'})\ \Phi_n^*(\mathbf{r'})\ dV'$$

For example, consider a point source at (r'', ϕ'', z'') in a cylindrical volume with radius a and height Z in cylindrical coordinates and $u(\mathbf{r'}) = u =$ constant (with $n \to nml$):

$$\rho(\mathbf{r'}, t') = u(\mathbf{r'})\ s(t') = u\ \delta(r'' - r')\ \delta(\phi'' - \phi')\ \delta(z'' - z')\ s(t')$$

$$\Phi_{nml}^*(\mathbf{r'}) = \sqrt{\frac{2}{\pi a^2 J_{n+1}^2(\beta_{nm}\ a)}}\ J_n(\beta_{nm}\ r')\ \cos(n\,\phi')\ \sqrt{\frac{2}{Z}}\ \sin\left(\frac{l\pi z'}{Z}\right)$$

where $J_n(\beta_{nm}\ r)$ is the n^{th} order Bessel function and $\beta_{nm}\ a$ is the m^{th} zero of J_n .

The volume integral then becomes:

$$\int_V u\ \delta(r'' - r')\ \delta(\phi'' - \phi')\ \delta(z'' - z')\ \sqrt{\frac{2}{\pi a^2 J_{n+1}^2(\beta_{nm}\ a)}}\ J_n(\beta_{nm}\ r')\ \cos(n\,\phi')\ \sqrt{\frac{2}{Z}}\ \sin\left(\frac{l\pi z'}{Z}\right) dV'$$

$$= u\, r'' \sqrt{\frac{2}{\pi a^2 J_{n+1}^2(\beta_{nm}\ a)}}\ J_n(\beta_{nm}\ r'')\ \cos(n\,\phi'')\ \sqrt{\frac{2}{Z}}\ \sin\left(\frac{l\pi z''}{Z}\right)$$

The solution now becomes:

$$\psi(\mathbf{r},t) = c^2 \frac{4u\, r''}{Z\, \pi\, a^2} \sum_{nml}{}' \left[\frac{1}{J_{n+1}^2(\beta_{nm}\, a)} J_n(\beta_{nm}\, r'') \cos(n\,\phi'') \sin\left(\frac{l\pi z''}{Z}\right) J_n(\beta_{nm}\, r) \cos(n\,\phi) \sin\left(\frac{l\pi z}{Z}\right) \int_{t'=-\infty}^{t} s(t') \frac{\sin(\omega_{nml}\,\tau)}{\omega_{nml}} dt' \right]$$

where we have replaced the limits of the time integral with $t_0 \rightarrow -\infty$ and $t+ \rightarrow t$ because s(t') will always be zero prior to time zero anyway, and t will always be greater than t'. We have also dropped $\theta(\tau)$ because t will always be greater than t' .

The modal frequencies for this case of a cylindrical volume are:

$$\omega_{nml} = \sqrt{(\beta_{nm}\, a)^2 \left(\frac{c}{a}\right)^2 + \left(\frac{l\,\pi\, z}{Z}\right)^2}$$

The resulting expression for the solution given above can be used for computation. For a fixed reception point at (x,y,z), an impulse response function can be precomputed for the impulse signal $s(t') = \delta(t')$.

Let

$$A_{nml} = c^2 \frac{4u\, r''}{Z\, \pi\, a^2} \frac{1}{J_{n+1}^2(\beta_{nm}\, a)} J_n(\beta_{nm}\, r'') \cos(n\,\phi'') \sin\left(\frac{l\pi z''}{Z}\right) J_n(\beta_{nm}\, r) \cos(n\,\phi) \sin\left(\frac{l\pi z}{Z}\right)$$

which can now be computed for all n, m, l. Development and use of the impulse response function defined by

$$f(t) = \sum_{nml}{}' \frac{A_{nml}}{\omega_{nml}} \sin(\omega_{nml}\, t)$$

is identical from this point on to that in [1].

References

[1] David R. Clark. Green's function method for creating accurate stereo sound images. EXE Consulting, P.O. Box 450998, Garland, Texas 75045- 0998, July 2004.

[2] Gabriel Barton. *Elements of Green's Functions and Propagation: Potentials, Diffusion, and Waves.* Oxford, 1989. Reprinted 1991.